

Some Results on Linear Codes over $\mathbb{Z}_4 + v\mathbb{Z}_4$

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Abstract

In this paper, we study the linear codes over the commutative ring $R = \mathbb{Z}_4 + v\mathbb{Z}_4$, where $v^2 = v$. We define the Gray weight of the elements of R and give a Gray map from R^n to \mathbb{Z}_4^{2n} , which lead to the MacWilliams identity of the linear code over R . Some useful results on self-dual code over R are given. Furthermore, the relationship between some complex unimodular lattices and Hermitian self-dual codes over R is given. Furthermore, the existing conditions of MDS codes over R is given, and the results show that there are no non-trivial MDS codes over R . Structural properties of cyclic codes are also discussed in this paper. As a special class of cyclic codes, quadratic residue and their extension codes over R are considered.

Keywords: MacWilliams identity; self-dual codes; unimodular lattices; cyclic codes; quadratic residue codes.

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1. Introduction

Error-Correcting and error-detecting codes play important roles in application ranging from data networking to satellite communication to compact disks. Most coding theory concerns on linear codes since they have clear structure that makes them simpler to discover, to understand and to encode and decode.

Codes over finite rings have been studied since the early 1970s. There are a lot of works on codes over finite rings after the discovery that certain good nonlinear binary codes can be constructed from cyclic codes over \mathbb{Z}_4 via the Gray map [7]. Since then, many researchers have paid more and more attentions to study the codes over finite rings. In these studies, the group rings associated with codes are finite chain rings.

Recently, Zhu et al. considered linear codes over the finite non-chain ring $\mathbb{F}_q + v\mathbb{F}_q$. In [13], they study the cyclic codes over $\mathbb{F}_2 + v\mathbb{F}_2$. It has shown that cyclic codes over this ring are principally generated. In the subsequent paper [14], they investigate a class of constacyclic codes over $\mathbb{F}_p + v\mathbb{F}_p$. In that paper, the authors prove that the image of a $(1-2v)$ -constacyclic code of length n over $\mathbb{F}_p + v\mathbb{F}_p$ under the Gray map is a cyclic code of length $2n$ over \mathbb{F}_p . Furthermore, they also assert that $(1-2v)$ -constacyclic codes over $\mathbb{F}_p + v\mathbb{F}_p$ are also principally generated.

Self-dual codes are an important class of linear codes. They have connections to many fields of research such as lattices, designs and invariant [1, 2]. The study of extremal self-dual codes has generated a lot of interests among the coding theory. And this is one of the motivations to introduce self-dual codes over the ring $\mathbb{Z}_4 + v\mathbb{Z}_4$.

As a special class of cyclic codes, quadratic residue codes fall into the family of BCH codes and have proven to be a promising family of cyclic codes. They were first introduced by Andrew

Gleason and since then have generated a lot of interests. This due to the fact that they enjoy good algebra properties and they contain source of good codes. Recently, quadratic residue codes have been studied over some special rings [9, 11].

In this paper, we mainly introduce some results on linear codes over the ring $R = \mathbb{Z}_4 + v\mathbb{Z}_4$. To the best of our knowledge, this is the first time to study the linear codes over this ring. The remainder of this paper is organized as follows. In Sec.2, we define the Gray weight of the element of R , and introduce a Gray map. This map leads to some useful results on linear codes over R . Moreover, we also give the MacWilliams identity on the linear code over R . In Sec.3, we introduce two important class of linear codes: self-dual codes and MDS codes. We give the sufficient and necessary conditions for a linear code to be self-dual and MDS. We also define the Hermitian dual on the linear code, and research the connections between Hermitian self-dual codes and complex lattices. Moreover, we obtain that there is no non-trivial MDS codes over R . In Sec.4, we study the cyclic codes over R including the generating polynomials, the generating idempotents and their duals. In Sec.5, we introduce an important class of cyclic codes called quadratic residue codes over R . Moreover, the extensions of quadratic residue codes are also discussed in this section. In Sec.6, we give some examples to illustrate the main results in this paper.

2. Linear codes over R

Let $R = \mathbb{Z}_4 + v\mathbb{Z}_4$, where $v^2 = v$. Then R is commutative and with characteristic 4. Clearly, $R \simeq \mathbb{Z}_4[v]/(v^2 - v)$. An element r of R can be expressed uniquely as $r = a + bv$, where $a, b \in \mathbb{Z}_4$. The ring R has the following properties

- There are 9 different ideals of R , and they are (1) , $(v+1)$, $(v+2)$, $(v-1)$, (2) , (v) , $(2v-2)$, $(2v)$, (0) ;
- R is a principal ring;
- $(v+1)$ and $(v+2)$ are the maximal ideals of R ;
- R is not a finite chain ring.

Furthermore, for any element $r = a + bv$ of R , r is a unit if and only if $a \not\equiv 0 \pmod{2}$ and $a + b \not\equiv 0 \pmod{2}$. Moreover, one can verify that if r is a unit of R then $r^2 = 1$.

Definition 1. Let $r = a + bv$ be any element of R . Then the Gray weight on R is a weight function defined as follows

$$w_G : R \rightarrow \mathbb{N}$$

$$a + bv \mapsto \begin{cases} 0, & \text{if } a = b = 0. \\ 1, & \text{if } a = 1, b = 3 \text{ or } a = 3, b = 1. \\ 1, & \text{if } a = 0, b = 1 \text{ or } a = 0, b = 3. \\ 2, & \text{if } a = b = 2 \text{ or } a = 0, b = 2. \\ 2, & \text{if } a = 1, b = 0 \text{ or } a = 1, b = 3. \\ 2, & \text{if } a = 3, b = 0 \text{ or } a = 3, b = 2. \\ 3, & \text{if } a = 1, b = 1 \text{ or } a = 3, b = 3. \\ 3, & \text{if } a = 2, b = 1 \text{ or } a = 2, b = 3. \\ 4, & \text{if } a = 2, b = 0. \end{cases}$$

Define a Gray weight of a vector $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in R^n$ to be the rational sum of the Gray weight of its components, i.e. $w_G(\mathbf{c}) = \sum_{i=0}^{n-1} w_G(c_i)$. For any elements $\mathbf{c}_1, \mathbf{c}_2 \in R^n$, the Gray distance is given by $d_G(\mathbf{c}_1, \mathbf{c}_2) = w_G(\mathbf{c}_1 - \mathbf{c}_2)$. A code C of length n over R is a subset of R^n . C is linear if and only if C is an R -submodule of R^n . The minimum Gray distance of C is the smallest nonzero Gray distance between all pairs of distinct codewords. The minimum Gray weight of C is the smallest nonzero Gray weight among all codewords. If C is linear, then the minimum Gray distance is the same as the minimum Gray weight.

Now we give the definition of the Gray map on R^n as follows

$$\Phi : R^n \rightarrow \mathbb{Z}_4^{2n}$$

$$(c_0, c_1, \dots, c_{n-1}) \mapsto (a_0, a_0 + b_0, \dots, a_{n-1}, a_{n-1} + b_{n-1}),$$

where $c_i = a_i + b_i v$, $i = 0, 1, \dots, n-1$.

It is well known that the Lee weights of the elements in \mathbb{Z}_4 are defined as $w_L(0) = 0$, $w_L(1) = w_L(3) = 1$ and $w_L(2) = 2$. Then we have the following result.

Theorem 1. The Gray map Φ is a distance-preserving map from R^n (Gray distance) to \mathbb{Z}_4^{2n} (Lee distance) and it is also \mathbb{Z}_4 -linear.

Proof. Let $k_1, k_2 \in \mathbb{Z}_4$. Then, by the definition of Gray map Φ , for any $\mathbf{c}_1, \mathbf{c}_2 \in R^n$ we have $\Phi(k_1 \mathbf{c}_1 + k_2 \mathbf{c}_2) = k_1 \Phi(\mathbf{c}_1) + k_2 \Phi(\mathbf{c}_2)$, which implies that Φ is \mathbb{Z}_4 -linear. Let $\mathbf{c}_1 = (c_{1,0}, c_{1,1}, \dots, c_{1,n-1})$ and $\mathbf{c}_2 = (c_{2,0}, c_{2,1}, \dots, c_{2,n-1})$ be elements of R^n , where $c_{1,i} = a_{1,i} + b_{1,i}v$ and $c_{2,i} = a_{2,i} + b_{2,i}v$, $i = 0, 1, \dots, n-1$. Then $\mathbf{c}_1 - \mathbf{c}_2 = (c_{1,0} - c_{2,0}, \dots, c_{1,n-1} - c_{2,n-1})$ and $\Phi(\mathbf{c}_1 - \mathbf{c}_2) = \Phi(\mathbf{c}_1) - \Phi(\mathbf{c}_2)$. Therefore $d_G(\mathbf{c}_1, \mathbf{c}_2) = w_G(\mathbf{c}_1 - \mathbf{c}_2) = w_L(\Phi(\mathbf{c}_1 - \mathbf{c}_2)) = w_L(\Phi(\mathbf{c}_1) - \Phi(\mathbf{c}_2)) = d_L(\Phi(\mathbf{c}_1), \Phi(\mathbf{c}_2))$. The second equality above holds because of the definition of the Gray weight of the element in R . \square

Lemma 1. Let C be a (n, M, d) linear code over R , where n, M, d are the length, the number of the codewords and the minimum Gray distance of C , respectively. Then $\Phi(C)$ is a $(2n, M, d)$ linear code over \mathbb{Z}_4 .

Proof. From Theorem 1, we see that $\Phi(C)$ is \mathbb{Z}_4 -linear, which implies that $\Phi(C)$ is a \mathbb{Z}_4 -linear code. From the definition of the Gray map Φ , $\Phi(C)$ is with length $2n$. Moreover, one can check that Φ is a bijective map from R^n to \mathbb{Z}_4^{2n} implying that $\Phi(C)$ has M codewords. At last, the preserving distance of Φ leads to C has the minimum Lee distance d . \square

Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ be two vectors of R^n . The Euclidean inner product of \mathbf{x} and \mathbf{y} is defined as follows

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i.$$

The dual code C^\perp of C is defined as $C^\perp = \{\mathbf{x} \in R^n | \mathbf{x} \cdot \mathbf{c} = 0 \text{ for all } \mathbf{c} \in C\}$. C is said to be self-orthogonal if $C \subseteq C^\perp$ and self-dual if $C = C^\perp$.

Theorem 2. Let C be a linear code. Then $\Phi(C)^\perp = \Phi(C^\perp)$. Moreover, if C is self-dual, so is $\Phi(C)$.

Proof. For all $\mathbf{c}_1 = (c_{1,0}, c_{1,1}, \dots, c_{1,n-1}) \in C$ and $\mathbf{c}_2 = (c_{2,0}, c_{2,1}, \dots, c_{2,n-1}) \in C^\perp$, where $c_{j,i} = a_{j,i} + b_{j,i}v$, $a_{j,i}, b_{j,i} \in \mathbb{Z}_4$, $j = 1, 2, i = 0, 1, \dots, n-1$, if $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$, then we have $\mathbf{c}_1 \cdot \mathbf{c}_2 = \sum_{i=0}^{n-1} c_{1,i} c_{2,i} = \sum_{i=0}^{n-1} a_{1,i} a_{2,i} + \sum_{i=0}^{n-1} (a_{1,i} b_{2,i} + a_{2,i} b_{1,i} + b_{1,i} b_{2,i})v = 0$ implying $\sum_{i=0}^{n-1} a_{1,i} a_{2,i} = 0$ and $\sum_{i=0}^{n-1} a_{1,i} b_{2,i} + a_{2,i} b_{1,i} + b_{1,i} b_{2,i} = 0$. Therefore, $\Phi(\mathbf{c}_1) \cdot \Phi(\mathbf{c}_2) = \sum_{i=0}^{n-1} a_{1,i} a_{2,i} + a_{1,i} a_{2,i} + a_{1,i} b_{2,i} + a_{2,i} b_{1,i} + b_{1,i} b_{2,i} = 0$. Thus $\Phi(C^\perp) \subseteq \Phi(C)^\perp$. From Lemma 1, we can verify that $|\Phi(C^\perp)| = |\Phi(C)^\perp|$, which implies that $\Phi(C)^\perp = \Phi(C^\perp)$. Clearly, $\Phi(C)$ is self-orthogonal if C is self-dual. From Lemma 1, we have $|\Phi(C)| = |C| = 16^{n/2} = 4^{2n/2}$. Thus, $\Phi(C)$ is self-dual. \square

One of the most remarkable results in coding theory is that the MacWilliams identity that describes the connections between a linear code and its dual on the weight enumerator. In the following, we discuss this issue over R .

Let C be a linear code of length n over R . Suppose that a is any element of R . For all $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in R^n$, define the weight of \mathbf{c} at a to be $w_a = |\{i | c_i = a\}|$.

Definition 2. Let A_i be the number of elements of the Gray weight i in C . Then the set $\{A_0, A_1, \dots, A_{4n}\}$ is called the Gray weight distribution of C . Define the Gray weight enumerator of C as $\text{Gray}_C(X, Y) = \sum_{i=0}^{4n} A_i X^{4n-i} Y^i$. Clearly, $\text{Gray}_C(X, Y) = \sum_{\mathbf{c} \in C} X^{4n-w_G(\mathbf{c})} Y^{w_G(\mathbf{c})}$. Furthermore, define the complete weight enumerator of C as $\text{cwe}_C(X_0, X_1, X_2, X_3, X_v, X_{1+v}, X_{2+v}, X_{3+v}, X_{2v}, X_{1+2v}, X_{2+2v}, X_{3+2v}, X_{3v}, X_{1+3v}, X_{2+3v}, X_{3+3v}) = \sum_{\mathbf{c} \in C} X_0^{w_0(\mathbf{c})} X_1^{w_1(\mathbf{c})} X_2^{w_2(\mathbf{c})} X_3^{w_3(\mathbf{c})} X_v^{w_v(\mathbf{c})} X_{1+v}^{w_{1+v}(\mathbf{c})} X_{2+v}^{w_{2+v}(\mathbf{c})} X_{3+v}^{w_{3+v}(\mathbf{c})} X_{2v}^{w_{2v}(\mathbf{c})} X_{1+2v}^{w_{1+2v}(\mathbf{c})} X_{2+2v}^{w_{2+2v}(\mathbf{c})} X_{3+2v}^{w_{3+2v}(\mathbf{c})} X_{3v}^{w_{3v}(\mathbf{c})} X_{1+3v}^{w_{1+3v}(\mathbf{c})} X_{2+3v}^{w_{2+3v}(\mathbf{c})} X_{3+3v}^{w_{3+3v}(\mathbf{c})}$.

For any codeword \mathbf{c} of C , let

$$\alpha_0(\mathbf{c}) = w_0(\mathbf{c})$$

$$\alpha_1(\mathbf{c}) = w_v(\mathbf{c}) + w_{3v}(\mathbf{c}) + w_{3+v}(\mathbf{c}) + w_{1+3v}(\mathbf{c})$$

$$\begin{aligned}\alpha_2(\mathbf{c}) &= w_1(\mathbf{c}) + w_3(\mathbf{c}) + w_{3v}(\mathbf{c}) + w_{1+2v}(\mathbf{c}) + w_{2+2v}(\mathbf{c}) + w_{3+2v}(\mathbf{c}) \\ \alpha_3(\mathbf{c}) &= w_{1+v}(\mathbf{c}) + w_{2+v}(\mathbf{c}) + w_{2+3v}(\mathbf{c}) + w_{3+3v}(\mathbf{c}) \\ \alpha_4(\mathbf{c}) &= w_2(\mathbf{c})\end{aligned}$$

denote the number of elements of \mathbf{c} with Gray weight 0, 1, 2, 3, 4, respectively. Then the Gray weight $w_G(\mathbf{c})$ of $\mathbf{c} \in C$ is defined to be

$$w_G(\mathbf{c}) = \alpha_1(\mathbf{c}) + 2\alpha_2(\mathbf{c}) + 3\alpha_3(\mathbf{c}) + 4\alpha_4(\mathbf{c}).$$

Define the symmetrized weight enumerator of C as $swe_C(X_0, X_1, X_2, X_3, X_4) = cwe_C(X_0, X_1, X_2, X_3, X_v, X_{1+v}, X_{2+v}, X_{3+v}, X_{2v}, X_{1+2v}, X_{2+2v}, X_{3v}, X_{1+3v}, X_{2+3v}, X_{3+3v}) = \sum_{\mathbf{c} \in C} X_0^{\alpha_0(\mathbf{c})} X_1^{\alpha_1(\mathbf{c})} X_2^{\alpha_2(\mathbf{c})} X_3^{\alpha_3(\mathbf{c})} X_4^{\alpha_4(\mathbf{c})}$. Furthermore, the Hamming weight enumerator of C is defined as

$$Ham_C(X, Y) = \sum_{\mathbf{c} \in C} X^{n-w_H(\mathbf{c})} Y^{w_H(\mathbf{c})},$$

where $w_H(\mathbf{c})$ denotes the Hamming weight of the codeword \mathbf{c} . Then we have the following results.

Theorem 3. Let C be a linear code of length n over R . Then

- (i) $Gray_C(X, Y) = swe_C(X^4, X^3Y, X^2Y^2, XY^3, Y^4)$;
- (ii) $Ham_C(X, Y) = swe_C(X, Y, Y, Y, Y)$;
- (iii) $Gray_C(X, Y) = Lee_{\Phi(C)}(X, Y)$;
- (iv) $Gray_{C^\perp}(X, Y) = \frac{1}{|C|} Gray_C(X + Y, X - Y)$;
- (v) $Ham_{C^\perp}(X, Y) = \frac{1}{|C|} Ham_C(X + 15Y, X - Y)$.

Proof. (i) From the definition of the symmetrized weight enumerator, we have

$$\begin{aligned}swe_C(X^4, X^3Y, X^2Y^2, XY^3, Y^4) &= \sum_{\mathbf{c} \in C} X^{4\alpha_0(\mathbf{c})} (X^3Y)^{\alpha_1(\mathbf{c})} (X^2Y^2)^{\alpha_2(\mathbf{c})} (XY^3)^{\alpha_3(\mathbf{c})} Y^{4\alpha_4(\mathbf{c})} \\ &= \sum_{\mathbf{c} \in C} X^{4\alpha_0(\mathbf{c})+3\alpha_1(\mathbf{c})+2\alpha_2(\mathbf{c})+\alpha_3(\mathbf{c})} Y^{\alpha_1(\mathbf{c})+2\alpha_2(\mathbf{c})+3\alpha_3(\mathbf{c})+4\alpha_4(\mathbf{c})} \\ &= \sum_{\mathbf{c} \in C} X^{4n-w_G(\mathbf{c})} Y^{w_G(\mathbf{c})} \\ &= Gray_C(X, Y).\end{aligned}$$

(ii) From the definition of symmetrized weight enumerator, we have

$$\begin{aligned}swe_C(X, Y, Y, Y, Y) &= \sum_{\mathbf{c} \in C} X^{\alpha_0(\mathbf{c})} Y^{\alpha_1(\mathbf{c})} Y^{\alpha_2(\mathbf{c})} Y^{\alpha_3(\mathbf{c})} Y^{\alpha_4(\mathbf{c})} \\ &= \sum_{\mathbf{c} \in C} X^{\alpha_0(\mathbf{c})} Y^{\alpha_1(\mathbf{c})+\alpha_2(\mathbf{c})+\alpha_3(\mathbf{c})+\alpha_4(\mathbf{c})} \\ &= \sum_{\mathbf{c} \in C} X^{n-w_H(\mathbf{c})} Y^{w_H(\mathbf{c})} \\ &= Ham_C(X, Y).\end{aligned}$$

(iii) From the definition of Gray weight enumerator, we obtain that

$$\begin{aligned}Gray_C(X, Y) &= \sum_{\mathbf{c} \in C} X^{4n-w_G(\mathbf{c})} Y^{w_G(\mathbf{c})} \\ &= \sum_{\Phi(\mathbf{c}) \in \Phi(C)} X^{4n-w_L(\Phi(\mathbf{c}))} Y^{w_L(\Phi(\mathbf{c}))} \\ &= Lee_{\Phi(C)}(X, Y).\end{aligned}$$

(iv) From Theorem 2, $\Phi(C^\perp) = \Phi(C)^\perp$ and they are both \mathbb{Z}_4 -linear according to Lemma 1. By Theorem 2.4 in [15] and (iii), we have

$$\begin{aligned}Gray_{C^\perp}(X, Y) &= Lee_{\Phi(C^\perp)}(X, Y) \\ &= Lee_{\Phi(C)^\perp}(X, Y) \\ &= \frac{1}{|\Phi(C)|} Lee_{\Phi(C)}(X + Y, X - Y) \\ &= \frac{1}{|C|} Gray_C(X + Y, X - Y).\end{aligned}$$

(v) It is straightforward from the Theorem 8.3 in [16]. \square

3. Self-dual and MDS codes

Self-dual codes are an important class of linear codes. They have been studied over a wide variety of rings, including finite fields, Galois rings and finite chain rings. Self-dual codes over rings have been shown to have closely interesting connections to the invariant theory, lattice theory and the theory of modular forms. In this section, we discuss the self-dual codes over the ring R . At the beginning, we introduce some useful facts.

By the Chinese Remainder Theorem, we have $R = vR \oplus (1-v)R = v\mathbb{Z}_4 \oplus (1-v)\mathbb{Z}_4$. Define

$$C_1 = \{\mathbf{x} \in \mathbb{Z}_4^n \mid \exists \mathbf{y} \in \mathbb{Z}_4^n, v\mathbf{x} + (1-v)\mathbf{y} \in C\}$$

and

$$C_2 = \{\mathbf{y} \in \mathbb{Z}_4^n \mid \exists \mathbf{x} \in \mathbb{Z}_4^n, v\mathbf{x} + (1-v)\mathbf{y} \in C\}.$$

Then C_1 and C_2 are both \mathbb{Z}_4 -linear of length n . Moreover, the linear code C of length n over R can be uniquely expressed as

$$C = vC_1 \oplus (1-v)C_2.$$

Theorem 4. Let C be a linear code of length n over R . Then $C^\perp = vC_1^\perp \oplus (1-v)C_2^\perp$. Moreover, C is self-dual if and only if C_1 and C_2 are both self-dual over \mathbb{Z}_4 .

Proof. Define

$$\widehat{C}_1 = \{\mathbf{x} \in \mathbb{Z}_4^n \mid \exists \mathbf{y} \in \mathbb{Z}_4^n, v\mathbf{x} + (1-v)\mathbf{y} \in C^\perp\}$$

and

$$\widehat{C}_2 = \{\mathbf{y} \in \mathbb{Z}_4^n \mid \exists \mathbf{x} \in \mathbb{Z}_4^n, v\mathbf{x} + (1-v)\mathbf{y} \in C^\perp\}.$$

Then $C^\perp = v\widehat{C}_1 + (1-v)\widehat{C}_2$ and this expression is unique. Clearly, $\widehat{C}_1 \subseteq C_1^\perp$. Let \mathbf{c}_1 be an element of C_1^\perp . Then, for any $\mathbf{x} \in C_1$, there exists $\mathbf{y} \in \mathbb{Z}_4^n$ such that $\mathbf{c}_1 \cdot (v\mathbf{x} + (1-v)\mathbf{y}) = 0$. Let $\mathbf{c} = v\mathbf{x} + (1-v)\mathbf{y} \in C$. Then $v\mathbf{c}_1 \cdot \mathbf{c} = 0$, which implies that $v\mathbf{c}_1 \in C^\perp$. By the unique expression of C^\perp , we have $\mathbf{c}_1 \in \widehat{C}_1$, i.e. $C_1 = \widehat{C}_1$. Similarly, we can prove $C_2 = \widehat{C}_2$ implying $C^\perp = vC_1^\perp + (1-v)C_2^\perp$.

Clearly, C is self-dual over R if C_1 and C_2 are both self-dual over \mathbb{Z}_4 . If C is self-dual, then C_1 and C_2 are both self-orthogonal over \mathbb{Z}_4 , i.e. $C_1 \subseteq C_1^\perp$ and $C_2 \subseteq C_2^\perp$. Next, we will prove $C_1 = C_1^\perp$ and $C_2 = C_2^\perp$. If not, then there are elements $\mathbf{a} \in C_1^\perp \setminus C_1$ and $\mathbf{b} \in C_2$ such that $(v\mathbf{a} + (1-v)\mathbf{b})^2 \neq 0$, which is a contradiction that C is self-dual. Therefore, $C_1 = C_1^\perp$ and $C_2 = C_2^\perp$. \square

For self-dual codes, the conditions of existing are very important for the enumeration. The following result is aim to resolve this issue.

Theorem 5. *There exist self-dual codes of any length n over R .*

Proof. Firstly, the element 2 of R generates a self-dual code of length 1 over R . Secondly, we assert that if C and \mathcal{D} are self-dual codes of length n and m over R respectively, then the direct product $C \times \mathcal{D}$ is also a self-dual code of length $n + m$ over R . In fact, let $(\mathbf{c}_1, \mathbf{d}_1), (\mathbf{c}_2, \mathbf{d}_2) \in C \times \mathcal{D}$. Then $(\mathbf{c}_1, \mathbf{d}_1) \cdot (\mathbf{c}_2, \mathbf{d}_2) = (\mathbf{c}_1 \cdot \mathbf{c}_2, \mathbf{d}_1 \cdot \mathbf{d}_2) = (\mathbf{0}, \mathbf{0})$, which implies that $C \times \mathcal{D}$ is self-orthogonal. Moreover, since C and \mathcal{D} are both self-dual over R , it follows that $|C| = |R|^{n/2}$ and $|\mathcal{D}| = |R|^{m/2}$. Therefore $|C \times \mathcal{D}| = |C||\mathcal{D}| = |R|^{(n+m)/2}$ implying $C \times \mathcal{D}$ is self-dual. \square

For a \mathbb{Z}_4 -linear code C , C and its dual C^\perp have G and G^\perp as their standard generator matrices, respectively

$$G = \begin{pmatrix} I_{k_1} & A & B \\ \mathbf{0} & 2I_{k_2} & 2C \end{pmatrix},$$

$$G^\perp = \begin{pmatrix} -B^t - C^t A^t & C^t & I_{n-k_1-k_2} \\ 2A^t & 2I_{k_2} & \mathbf{0} \end{pmatrix}.$$

Furthermore, C and C^\perp are of the type $4^{k_1}2^{k_2}$ and $4^{n-k_1-k_2}2^{k_2}$, respectively. Therefore C is self-dual over \mathbb{Z}_4 if and only if C and C^\perp are of the same type, which implies that C is of type 4^k2^{n-2k} . Then, by Theorem 4 and Theorem 5 and Theorem 12.5.7 in [8], we have the following straightforward result.

Theorem 6. *For $0 \leq k \leq \lfloor n/2 \rfloor$, there are $v_{n,k}2^{k(k+1)}$ self-dual codes over R of length n , where $v_{n,k}$ is the number of $[n, k]$ self-orthogonal doubly-even (i.e. the Hamming weight of every codeword is divisible by 4) binary codes. The total number of self-dual code over R of length n is*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} v_{n,k} 2^{k(k+1)}.$$

In the following of this section, we discuss some special class of self-dual codes over R . It needs the following definition first.

Definition 3. *Let $r = a + vb$ be an element of R . Then the Euclidean weight of r is defined as follows*

$$w_E(r) = w_E(a) + w_E(a + b),$$

where

$$w_E(a) = \min\{|a|^2, |4 - a|^2\}$$

and

$$w_E(a + b) = \min\{|a + b|^2, |4 - a - b|^2\}.$$

The Euclidean weight of a vector $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in R^n$ is the rational sum of the Euclidean weight of its components, i.e. $w_E(\mathbf{c}) = \sum_{i=0}^{n-1} w_E(c_i)$.

Lemma 2. *The Gray map Φ is Euclidean weight-preserving from R^n to \mathbb{Z}_4^{2n} .*

Proof. It is well known that the Euclidean weight of the element a of \mathbb{Z}_4 is defined as $w_E(a) = \min\{|a|^2, |4 - a|^2\}$ and the Euclidean weight of a vector $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in \mathbb{Z}_4^n$ is the rational sum of the Euclidean weight of its components, i.e. $w_E(\mathbf{c}) = \sum_{i=0}^{n-1} w_E(c_i)$. Then, by the definitions of the Gray map Φ and the Euclidean weight of the element of R , we can show that Φ is a Euclidean weight-preserving map from R^n to \mathbb{Z}_4^{2n} . \square

A self-dual code C of length n over R is called Type II if the Euclidean weight of every codeword of C is multiple of 8, otherwise C is called Type I.

Theorem 7. *Let C be a self-dual code of length n over R . Then*

(i) *C is Type II if and only if n is multiple of 4.*

(ii) *If C is Type II, so is $\Phi(C)$.*

(iii) *The minimum Euclidean weight of C satisfies*

$$d_E \leq 8\lfloor n/12 \rfloor + 8.$$

Proof. Let C be a self-dual code of length n over R . Then, by Theorem 2, $\Phi(C)$ is a self-dual code of length $2n$ over \mathbb{Z}_4 . From Lemma 2, we have that (ii) is valid. For (i), it is well known that there exist self-dual codes of length n over \mathbb{Z}_4 if and only if n is multiple of 8 [1], which follows (i). (iii) is follows from the Theorem 12.5.1 of [8]. \square

The self-dual codes meeting the bound in Theorem 7(iii) are called extremal. By Lemma 2 and Theorem 7(ii), if C is an extremal Type II code over R , so is $\Phi(C)$ over \mathbb{Z}_4 . The bound of Theorem 7(iii) is obviously the bound on the minimum Gray weight of self-dual codes over R , but highly unsatisfactory one.

By the Theorem 12.5.8 in [8], we have the following result on the enumerator of Type II codes over R .

Theorem 8. *Let $n \equiv 0 \pmod{4}$ and $N = 2n$. Then there are $\mu_{N,k}2^{1+k(k-1)/2}$ Type II codes of length n over R for $0 \leq k \leq n$, where $\mu_{N,k}$ is the number of $[N, k]$ self-orthogonal doubly-even binary codes containing the vector $\mathbf{1}$. The total number of Type II codes of length n is*

$$\sum_{k=0}^n \mu_{N,k} 2^{1+k(k-1)/2}.$$

At the beginning of this section, we assert that self-dual codes are connected with the lattice theory. We will discuss how to construct the complex lattice from the self-dual code over R . Firstly, we need another inner product on R^n called Hermitian inner-product.

Definition 4. *Let $\mathbf{w}, \mathbf{u} \in R^n$. Then the Hermitian inner-product of \mathbf{w}, \mathbf{u} is defined as $\langle \mathbf{w}, \mathbf{u} \rangle = \sum_{i=0}^{n-1} w_i \bar{u}_i$, where $\bar{v} = 1 - v$. For any code C of length n over R , the Hermitian dual of C is $C^H = \{\mathbf{w} \in R^n \mid \langle \mathbf{w}, \mathbf{u} \rangle = 0, \forall \mathbf{u} \in C\}$.*

Let $2\ell + 1$ be a square free integer with $\ell \equiv 7 \pmod{8}$. Define $K = \mathbb{Q}(\sqrt{-2\ell - 1})$. Let $\omega = \frac{1 + \sqrt{-2\ell - 1}}{2}$. Define $\mathcal{O}_K = \mathbb{Z}[\omega]$, we have that \mathcal{O}_K is the ring of integers of the field K . The ω satisfies the equation $X^2 - X + \frac{\ell+1}{2}$. Notice that $\ell \equiv 7 \pmod{8}$ so that $\frac{\ell+1}{2}$ is an integer divisible by 4.

Consider the canonical homomorphism $\rho: \mathcal{O}_K \rightarrow \mathcal{O}_K/4\mathcal{O}_K$. Now the image of ω satisfies the equation $X^2 - X = 0$.

Lemma 3. *The ring $O_K/4O_K$ is ring isomorphic to R .*

Proof. Define the map $\Psi : O_K/4O_K \rightarrow R$ by $\Psi(a+b\omega) = a+bv$, where $a, b \in \mathbb{Z}_4$. The map is bijective and the fact that it is a homomorphism follows that $\omega^2 = \omega$ in $O_K/4O_K$. \square

Furthermore, we notice that $\Psi(\overline{\omega}) = \overline{\Psi(\frac{1}{2} + \frac{\sqrt{-2\ell-1}}{2})} = \Psi(\frac{1}{2} + \frac{\sqrt{2\ell+1}}{2}i) = \Psi(\frac{1}{2} - \frac{\sqrt{2\ell+1}}{2}i) = 1 - v = \overline{v}$. Therefore complex conjugation corresponds to conjugation in R via the isomorphism Ψ .

A lattice Λ over K is an O_K -submodule of K^n with full rank. The Hermitian dual of Λ is defined as

$$\Lambda^* = \{\mathbf{v} \in K^n \mid \langle \mathbf{v}, \mathbf{w} \rangle \in O_K, \forall \mathbf{w} \in \Lambda\}.$$

If $\Lambda = \Lambda^*$, we say Λ is unimodular and if $\Lambda \subseteq \Lambda^*$, we say Λ is integral.

Lemma 4. *Let C be a linear code of length n over R . Then we have the following results*

- (i) $\Lambda(C) = \{\mathbf{v} \in O_K^n \mid \rho(\mathbf{v}) \in C\}$ is an O_K -lattice.
- (ii) $\Lambda(C^H) = 4\Lambda(C)^*$.
- (iii) $(\frac{1}{2}\Lambda(C))^* = 2\Lambda(C)^*$.

Proof. (i) It is immediate from the definition of $\Lambda(C)$ and C is an R -submodule of R^n .

(ii) If $\mathbf{v} \in 4\Lambda(C)^*$, then $\langle \frac{1}{4}\mathbf{v}, \mathbf{w} \rangle \in O_K$ for all $\mathbf{w} \in \Lambda(C)$. Therefore, we have $\sum_{i=0}^{n-1} \frac{1}{4}v_i\overline{w}_i \in O_K \Rightarrow \sum_{i=0}^{n-1} v_i\overline{w}_i \in 4O_K \Rightarrow \langle \rho(\mathbf{v}), \rho(\mathbf{w}) \rangle = 0$, which implies that $\mathbf{v} \in \Lambda(C^H)$. Then $4\Lambda(C) \subseteq \Lambda(C^H)$.

Let $\mathbf{v} \in \Lambda(C^H)$. Then $\rho(\mathbf{v}) \in C^H$ and $\langle \rho(\mathbf{v}), \rho(\mathbf{w}) \rangle = 0$ for all $\mathbf{w} \in \Lambda(C)$. Then we have $\sum_{i=0}^{n-1} v_i\overline{w}_i \in 4O_K \Rightarrow \sum_{i=0}^{n-1} \frac{1}{4}v_i\overline{w}_i \in O_K \Rightarrow \langle \frac{1}{4}\mathbf{v}, \mathbf{w} \rangle \in O_K$, which implies that $\mathbf{v} \in 4\Lambda(C)^*$. Therefore $\Lambda(C^H) = 4\Lambda(C)^*$.

(iii) Let $\mathbf{v} \in (\frac{1}{2}\Lambda(C))^*$, that is $\langle \mathbf{v}, \mathbf{w} \rangle \in O_K$ for all $\mathbf{w} \in \frac{1}{2}\Lambda(C)$. This implies that $(\frac{1}{2} \times 2)\langle \mathbf{v}, \mathbf{w} \rangle \in O_K$. Then we have $\langle \frac{1}{2}\mathbf{v}, 2\mathbf{w} \rangle \in O_K$ for all $\mathbf{w} \in \frac{1}{2}\Lambda(C)$, that is for all $2\mathbf{w} \in \Lambda(C)$. Then we have $\frac{1}{2}\mathbf{v} \in \Lambda(C)^*$, which implies that $\mathbf{v} \in 2\Lambda(C)^*$. Therefore $(\frac{1}{2}\Lambda(C))^* \subseteq 2\Lambda(C)^*$.

Now, assume that $\mathbf{v} \in 2\Lambda(C)^*$. Then $\langle \mathbf{v}, \mathbf{w} \rangle \in O_K$ for all $\mathbf{w} \in 2\Lambda(C)$. That is $\frac{1}{2}\langle \mathbf{v}, \mathbf{w} \rangle \in O_K$ for all $\mathbf{w} \in \Lambda(C)$, which implies that $\mathbf{v} \in (\frac{1}{2}\Lambda(C))^*$. Then $2\Lambda(C)^* = (\frac{1}{2}\Lambda(C))^*$. \square

Theorem 9. *The code C over R is Hermitian self-dual if and only if $\frac{1}{2}\Lambda(C)$ is unimodular.*

Proof. If $C = C^H$, then by Lemma 4(iii), $(\frac{1}{2}\Lambda(C))^* = 2\Lambda(C)^*$. Furthermore, by Lemma 4(ii), we have $\Lambda(C^H) = 4\Lambda(C)^*$, which implies that $2\Lambda(C)^* = \frac{1}{2}\Lambda(C^H) = \frac{1}{2}\Lambda(C)$. Therefore $\frac{1}{2}\Lambda(C) = (\frac{1}{2}\Lambda(C))^*$.

Next, let $\frac{1}{2}\Lambda(C) = (\frac{1}{2}\Lambda(C))^*$. Then $(\frac{1}{2}\Lambda(C))^* = 2\Lambda(C)^*$ by Lemma 4(iii). Furthermore, $2\Lambda(C)^* = \frac{1}{2}\Lambda(C^H)$ by Lemma 4(ii). Therefore, we have $\frac{1}{2}\Lambda(C^H) = \frac{1}{2}\Lambda(C)$. In the following, we show $C = C^H$. Let $\mathbf{v} \in C$. Then there exists $\mathbf{w} \in \Lambda(C)$ such that $\rho(\mathbf{w}) = \mathbf{v}$. But $\Lambda(C) = \Lambda(C^H)$, which implies that $\rho(\mathbf{w}) \in C^H$. This yields $C \subseteq C^H$. Similarly, we can prove $C^H \subseteq C$. Thus $C = C^H$ implying C is Hermitian self-dual. \square

In the rest of this section, we discuss another class of useful linear codes over R called MDS codes. For any Frobenius ring R , the Singleton bound for a code of length n over R states that

$$d_H(C) \leq n - \log_{|R|} |C| + 1,$$

where $d_H(C)$ denotes the minimum Hamming distance of C . A code meeting this bound is said to be an MDS code over R .

Theorem 10. *Let $C = vC_1 \oplus (1-v)C_2$ be a linear code of length n over R . Then we have*

- (i) $d_H(C) = \min\{d_H(C_1), d_H(C_2)\}$;
- (ii) C is an (n, M, d) MDS code over R if and only if C_1 and C_2 are both (n, \sqrt{M}, d) MDS code over \mathbb{Z}_4 .

Proof. (i) It is straightforward from the fact that for any codeword $\mathbf{c} = v\mathbf{c}_1 + (1-v)\mathbf{c}_2 \in C$, $\mathbf{c} = \mathbf{0}$ if and only if $\mathbf{c}_1 = \mathbf{c}_2 = \mathbf{0}$.

(ii) Denote $d_H^{(1)}(C)$ and $d_H^{(2)}(C)$ as the minimum Hamming distances of C_1 and C_2 , respectively. If $d_H(C) = d_H^{(1)}(C)$, then $d_H^{(2)}(C) \geq d_H^{(1)}(C)$ by (i). Let C be an (n, M, d) MDS code. Then $d = n - \log_{16} |C| + 1$. Let M_1 and M_2 are the codewords number of C_1 and C_2 , respectively. Then, by the Singleton bound, we have

$$d_H^{(1)} \leq n - \log_4 M_1 + 1$$

and

$$d_H^{(2)} \leq n - \log_4 M_2 + 1.$$

From $d = d_H^{(1)} = d_H^{(2)}$, we have that

$$\log_4 \sqrt{M} \geq \log_4 M_1 \tag{1}$$

and

$$\log_4 \sqrt{M} \geq \log_4 M_2. \tag{2}$$

Therefore the equality in the above equations (1) and (2) hold if and only if $M_1 = M_2 = \sqrt{M}$. By the Singleton bound and C is an MDS code, we deduce C_1 and C_2 are both MDS code with the same parameters. The necessary part is straightforward by the Singleton bound. \square

Corollary 1. *There are no non-trivial MDS codes over R .*

Proof. By Theorem 10(ii), we know that there exist non-trivial MDS codes over R if and only if there exist non-trivial MDS codes over \mathbb{Z}_4 . But it is well known that there are no non-trivial MDS codes over \mathbb{Z}_4 anymore. \square

4. Cyclic codes over R

As a special class of linear codes, cyclic codes play a very important role in the coding theory. In this section, we give some useful results on cyclic codes over R .

Let T be a cyclic shift operator on R^n , i.e. for any vector $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in R^n$, $T(\mathbf{c}) = (c_{n-1}, c_0, \dots, c_{n-2})$.

A linear code C of length n over R is called cyclic if and only if $T(C) = C$. Define the polynomial ring $R_n = R[X]/(X^n - 1) = \{c_0 + c_1X + \dots + c_{n-1}X^{n-1} + (X^n - 1) \mid c_0, c_1, \dots, c_{n-1} \in R\}$. For any polynomial $c(X) + (X^n - 1) \in R_n$, we denote it as $c(X)$ for simplicity.

Define a map as follows

$$\begin{aligned}\varphi: R^n &\rightarrow R_n = R[X]/(X^n - 1) \\ (c_0, c_1, \dots, c_{n-1}) &\mapsto c(X) = c_0 + c_1X + \dots + c_{n-1}X^{n-1}.\end{aligned}$$

Clearly, φ is an R -module isomorphism from R^n to R_n . And a linear code C of length n is cyclic over R if and only if $\varphi(C)$ is an ideal of R_n . Sometimes, we identify the cyclic code C to the ideal of R_n .

Theorem 11. *A linear code $C = vC_1 \oplus (1-v)C_2$ is cyclic over R if and only if C_1 and C_2 are both cyclic over \mathbb{Z}_4 .*

Proof. Let $(a_0, a_1, \dots, a_{n-1}) \in C_1$ and $(b_0, b_1, \dots, b_{n-1}) \in C_2$. Assume that $c_i = va_i + (1-v)b_i$ for $i = 0, 1, \dots, n-1$. Then the vector $(c_0, c_1, \dots, c_{n-1})$ belongs to C . Since C is a cyclic code, it follows that $(c_{n-1}, c_0, \dots, c_{n-2}) \in C$. Note that $(c_{n-1}, c_0, \dots, c_{n-2}) = v(a_{n-1}, a_0, \dots, a_{n-2}) + (1-v)(b_{n-1}, b_0, \dots, b_{n-2})$. Hence $(a_{n-1}, a_0, \dots, a_{n-1}) \in C_1$ and $(b_{n-1}, b_0, \dots, b_{n-2}) \in C_2$, which implies that C_1 and C_2 are both cyclic codes over \mathbb{Z}_4 .

Conversely, suppose that C_1 and C_2 are both cyclic codes over \mathbb{Z}_4 . Let $(c_0, c_1, \dots, c_{n-1}) \in C$, where $c_i = va_i + (1-v)b_i$ for $i = 0, 1, \dots, n-1$. Then $(a_0, a_1, \dots, a_{n-1}) \in C_1$ and $(b_0, b_1, \dots, b_{n-1}) \in C_2$. Note that $(c_{n-1}, c_0, \dots, c_{n-2}) = v(a_{n-1}, a_0, \dots, a_{n-2}) + (1-v)(b_{n-1}, b_0, \dots, b_{n-2}) \in vC_1 \oplus (1-v)C_2 = C$. Therefore, C is a cyclic code over R . \square

In the following of this section, we assume that n is an odd positive integer. Let C be a cyclic code of length n over \mathbb{Z}_4 . Then there exist unique monic polynomials $f(X), g(X), h(X)$ such that $X^n - 1 = f(X)g(X)h(X)$ and $C = (f(X)g(X)) \oplus (2f(X)h(X))$. See [15] for the details.

Theorem 12. *Let $C = vC_1 \oplus (1-v)C_2$ be a cyclic code of length n over R . Then $C = (vf_1(X)g_1(X) + (1-v)f_2(X)g_2(X)) \oplus (2vf_1(X)h_1(X) + 2(1-v)f_2(X)h_2(X))$, where $f_1(X)g_1(X)h_1(X) = f_2(X)g_2(X)h_2(X) = X^n - 1$ and $C_1 = (f_1(X)g_1(X)) \oplus (2f_1(X)h_1(X))$, $C_2 = (f_2(X)g_2(X)) \oplus (2f_2(X)h_2(X))$ over \mathbb{Z}_4 , respectively.*

Proof. Let $\tilde{C} = (vf_1(X)g_1(X) + (1-v)f_2(X)g_2(X)) \oplus (2vf_1(X)h_1(X) + 2(1-v)f_2(X)h_2(X))$, $C_1 = (f_1(X)g_1(X)) \oplus (2f_1(X)h_1(X))$ and $C_2 = (f_2(X)g_2(X)) \oplus (2f_2(X)h_2(X))$. Clearly, $\tilde{C} \subseteq C$. For vC_1 , we have $vC_1 = vC$ since $v^2 = v$ over \mathbb{Z}_4 . Similarly, $(1-v)C_2 = (1-v)C$. Therefore $vC_1 \oplus (1-v)C_2 \subseteq C$. Thus $C = \tilde{C}$. \square

Corollary 2. *The quotient polynomial ring $R[X]/(X^n - 1)$ is principal.*

Proof. Let $C = (f(X)g(X)) \oplus (2f(X)h(X))$ be a cyclic code of length n over \mathbb{Z}_4 , where $X^n - 1 = f(X)g(X)h(X)$. Then $C = (f(X)g(X) + 2f(X))$. (See Theorem 7.25 and Theorem 7.26 in [15] for the details.) By Theorem 12, we have any cyclic code C is principal over R , which implies the results. \square

Furthermore, the number of distinct cyclic codes of odd length n over R is 9^r , where r is the number of the basic irreducible factors of $X^n - 1$ over \mathbb{Z}_4 .

We have observed numerous times that self-dual cyclic codes over R exist. (See Example xx in Section 6.) Theorem 12 gives the generating polynomials for cyclic codes over R . The next result gives the conditions on these polynomials that lead to self-dual codes.

Theorem 13. *Let $C = (vf_1(X)g_1(X) + (v-1)f_2(X)g_2(X)) \oplus (2vf_1(X)h_1(X) + 2(v-1)f_2(X)h_2(X))$, where $f_1(X)g_1(X)h_1(X) = f_2(X)g_2(X)h_2(X) = X^n - 1$ and $C_1 = (f_1(X)g_1(X)) \oplus (2f_1(X)h_1(X))$, $C_2 = (f_2(X)g_2(X)) \oplus (2f_2(X)h_2(X))$ over \mathbb{Z}_4 , respectively. Then C is self-dual if and only if $f_1(X) = h_1^*(X)$, $g_1(X) = g_1^*(X)$ and $f_2(X) = h_2^*(X)$, $g_2(X) = g_2^*(X)$, where $f^*(X) = X^{\deg f(X)}f(X^{-1})$.*

Proof. Firstly, by $C^\perp = vC_1^\perp \oplus (v-1)C_2^\perp$, we have C^\perp is also a cyclic code if C is a cyclic code. Moreover, by Theorem 4, we have C is self-dual over R if and only if C_1 and C_2 are both self-dual over \mathbb{Z}_4 . Then, by Theorem 12.5.10 in [8], we deduce the result. \square

When do there exist non-zero cyclic self-dual codes of odd length n over R ? By Theorem 4 and Theorem 3 in [10], we give an answer on this problem.

Theorem 14. *Non-zero cyclic self-dual codes of odd length n exist over R if and only if $-1 \not\equiv 2^j \pmod{n}$ for any j .*

For example, if $n = 7$, then n satisfies the condition in Theorem 14. And then, there exists non-zero self-dual codes of length 7 over R . The Example xx in Section 6 shows that there exist non-zero self-dual codes of length 7 over R indeed.

An element $e(X) \in C$ is called an idempotent element if $e(X)^2 = e(X)$ in R_n .

Theorem 15. *Let C be a cyclic code of odd length n . Then there exists a unique idempotent element $e(X) = ve_1(X) + (1-v)e_2(X) \in R[X]$ such that $C = (e(X))$.*

Proof. If n is odd, then there exist unique idempotent elements $e_1(X), e_2(X) \in \mathbb{Z}_4[X]$ such that $C_1 = (e_1(X))$ and $C_2 = (e_2(X))$. By Theorem 12, we have $C = (ve_1(X) + (1-v)e_2(X))$. Let $e(X) = ve_1(X) + (1-v)e_2(X)$. Then $e(X)^2 = ve_1(X)^2 + (1-v)e_2(X)^2 = ve_1(X) + (1-v)e_2(X) = e(X)$, which implies that $e(X)$ is an idempotent element of C . If there is another $d(X) \in C$ such that $C = (d(X))$ and $d(X)^2 = d(X)$. Since $d(X) \in C = (e(X))$, we have that $d(X) = a(X)e(X)$ for some $a(X) \in R_n$. And then, $d(X)e(X) = a(X)e(X)^2 = d(X)$. Similarly, we can prove $d(X)e(X) = e(X)$, which implies that $e(X)$ is unique. \square

The idempotent element $e(X)$ in above Theorem is called the generating idempotent of C .

Theorem 16. *Let $C = vC_1 \oplus (1-v)C_2$ be a cyclic code of length n over R . Let $e(X) = ve_1(X) + (1-v)e_2(X)$, where $e_1(X)$ and $e_2(X)$ are generating idempotents of C_1 and C_2 over \mathbb{Z}_4 , respectively. Then the dual code C^\perp has $1 - e(X^{-1})$ as its generating idempotent.*

Proof. By Theorem 4, we have $C^\perp = vC_1^\perp \oplus (v-1)C_2^\perp$. Moreover, C^\perp is also a cyclic code since C_1^\perp and C_2^\perp are both cyclic

codes. Let $e_1(X)$ and $e_2(X)$ be generating idempotents of C_1 and C_2 , respectively. Then C_1^\perp and C_2^\perp have $1 - e_1(X^{-1})$ and $1 - e_2(X^{-1})$ as their generating idempotents respectively. (See Lemma 12.3.23(i) in [8] for the details.) Let $\tilde{e}(X)$ be the generating idempotent of C^\perp . Then, by Theorem 15, $\tilde{e}(X) = v(1 - e_1(X^{-1})) + (1 - v)(1 - e_2(X^{-1})) = 1 - e(X^{-1})$. \square

5. Quadratic residue codes over R

In this section, let p be a prime number with $p \equiv \pm 1 \pmod{8}$. Let Q_p denote the set of nonzero quadratic residues modulo p , and let N_p be the set of quadratic non-residues modulo p .

Let $Q(X) = \sum_{i \in Q_p} X^i$, $N(X) = \sum_{i \in N_p} X^i$ and $J(X) = p \sum_{i=0}^{p-1} X^i$. By Theorem 15 and Theorem 8 in [11], we have the following results immediately.

Lemma 5. Define r by $p = 8r \pm 1$. If r is odd, denote the set $S_0 = \{Q(X) + 2N(X), N(X) + 2Q(X), 1 - Q(X) + 2N(X), 1 - N(X) + 2Q(X)\}$. If r is even, denote the set $S_e = \{-Q(X), -N(X), 1 + Q(X), 1 + N(X)\}$. Then

- (i) For any $e_i(X), e_2(X) \in S_0$ or $e_i(X), e_2(X) \in S_e$, we have $e(X) = ve_1(X) + (1 - v)e_2(X)$ is the idempotent of R_p .
- (ii) $J(X)$ is an idempotent of R_p .

We now discuss the quadratic residue codes over R . Firstly, we give the definitions of these codes. The definitions depend upon the value p modulo 8.

Case I: $p \equiv -1 \pmod{8}$

Definition 5. Let $p + 1 = 8r$. If r is odd, define

$$\mathcal{D}_1 = (v(Q(X) + 2N(X)) + (1 - v)(N(X) + 2Q(X))),$$

$$\mathcal{D}_2 = (v(N(X) + 2Q(X)) + (1 - v)(Q(X) + 2N(X))),$$

and

$$\mathcal{E}_1 = (v(1 - N(X) + 2Q(X)) + (1 - v)(1 - Q(X) + 2N(X))),$$

$$\mathcal{E}_2 = (v(1 - Q(X) + 2N(X)) + (1 - v)(1 - N(X) + 2Q(X))).$$

If r is even, define

$$\mathcal{D}_1 = (v(-Q(X)) + (1 - v)(-N(X))),$$

$$\mathcal{D}_2 = (v(-N(X)) + (1 - v)(-Q(X))),$$

and

$$\mathcal{E}_1 = (v(1 + N(X)) + (1 - v)(1 + Q(X))),$$

$$\mathcal{E}_2 = (v(1 + Q(X)) + (1 - v)(1 + N(X))).$$

These eight cyclic codes of length p are called the quadratic residue codes over R at the case I.

Let a be a non-zero positive integer defined as $\mu_a(i) = ai$ for any positive integer i . This map acts on polynomials as

$$\mu_a\left(\sum_i X^i\right) = \sum_i X^{ai}.$$

Theorem 17. Let $p \equiv -1 \pmod{8}$. Then the quadratic residue codes defined above satisfy the following:

- (i) $\mathcal{D}_i\mu_a = \mathcal{D}_i$ and $\mathcal{E}_i\mu_a = \mathcal{E}_i$ for $i = 1, 2$ and $a \in Q_p$; $\mathcal{D}_1\mu_a = \mathcal{D}_2$ and $\mathcal{E}_1\mu_a = \mathcal{E}_2$ for $a \in N_p$.
- (ii) $\mathcal{D}_1 \cap \mathcal{D}_2 = (J(X))$ and $\mathcal{D}_1 + \mathcal{D}_2 = R_p$.
- (iii) $\mathcal{E}_1 \cap \mathcal{E}_2 = \{0\}$ and $\mathcal{E}_1 \cap \mathcal{E}_2 = (J(X))^\perp$.
- (iv) $|\mathcal{D}_1| = |\mathcal{D}_2| = 4^{p+1}$ and $|\mathcal{E}_1| = |\mathcal{E}_2| = 4^{p-1}$.
- (v) $\mathcal{D}_i = \mathcal{E}_i + (J(X))$ for $i = 1, 2$.
- (vi) \mathcal{E}_1 and \mathcal{E}_2 are self-orthogonal and $\mathcal{E}_i^\perp = \mathcal{D}_i$ for $i = 1, 2$.

Proof. Let $p + 1 = 8r$. We only verify when r is odd. The case of r is even can be proved similarly.

- (i) If $a \in Q_p$, then $(v(Q(X) + 2N(X)) + (1 - v)(N(X) + 2Q(X)))\mu_a = v(Q(X) + 2N(X)) + (1 - v)(N(X) + 2Q(X))$, which implies that $\mathcal{D}_1\mu_a = \mathcal{D}_1$. Similarly, $\mathcal{D}_2\mu_a = \mathcal{D}_2$.

If $a \in N_p$, then $(v(Q(X) + 2N(X)) + (1 - v)(N(X) + 2Q(X)))\mu_a = v(N(X) + 2Q(X)) + (1 - v)(Q(X) + 2N(X))$, which implies that $\mathcal{D}_1\mu_a = \mathcal{D}_2$.

The parts of (i) involving \mathcal{E}_i are similar.

- (ii) Since $p \equiv -1 \pmod{8}$, it follows that $J(X) = 3 \sum_{i=0}^{p-1} X^i = 3 + 3Q(X) + 3N(X)$. Therefore $(v(Q(X) + 2N(X)) + (1 - v)(N(X) + 2Q(X)))(v(N(X) + 2Q(X)) + (1 - v)(Q(X) + 2N(X))) = (Q(X) + 2N(X))(N(X) + 2Q(X)) = J(X)$, which implies that $\mathcal{D}_1 \cap \mathcal{D}_2 = (J(X))$. Moreover, $v(Q(X) + 2N(X)) + (1 - v)(N(X) + 2Q(X)) + v(N(X) + 2Q(X)) + (1 - v)(Q(X) + 2N(X)) - J(X) = 3Q(X) + 3N(X) - J(X) = 1$, which implies that $\mathcal{D}_1 + \mathcal{D}_2 = R_p$.

- (iii) For $\mathcal{E}_1 \cap \mathcal{E}_2$, we have $(v(1 - N(X) + 2Q(X)) + (1 - v)(1 - Q(X) + 2N(X)))(v(1 - Q(X) + 2N(X)) + (1 - v)(1 - N(X) + 2Q(X))) = (1 - N(X) + 2Q(X))(1 - Q(X) + 2N(X)) = 1 + N(X) + Q(X) + J(X) = 0$, which implies that $\mathcal{E}_1 \cap \mathcal{E}_2 = \{0\}$.

For $\mathcal{E}_1 + \mathcal{E}_2$, it has generating idempotent $1 - N(X) + 2Q(X) + 1 - Q(X) + 2N(X) = 2 + N(X) + Q(X) = 1 - J(X) = 1 - J(X)\mu_{-1}$ as $j(X)\mu_{-1} = J(X)$. Then, by Theorem 16, $\mathcal{E}_1 + \mathcal{E}_2 = (J(X))^\perp$.

- (iv) We use the fact that $|\mathcal{D}_1 + \mathcal{D}_2| = |\mathcal{D}_1||\mathcal{D}_2|/|\mathcal{D}_1 \cap \mathcal{D}_2|$. By (i), $|\mathcal{D}_1| = |\mathcal{D}_2|$, and by (ii), $|\mathcal{D}_1 + \mathcal{D}_2| = 16^p$ and $|\mathcal{D}_1 \cap \mathcal{D}_2| = 16$. Therefore, $|\mathcal{D}_1| = |\mathcal{D}_2| = 16^{(p+1)/2} = 4^{p+1}$. Similarly, by (i) and (iii), we can prove $|\mathcal{E}_1| = |\mathcal{E}_2| = 4^{p-1}$.

- (v) From (ii), we have $J(X) \in \mathcal{D}_2$ implying that $(v(N(X) + 2Q(X)) + (1 - v)(Q(X) + 2N(X)))J(X) = J(X)$ as $v(N(X) + 2Q(X)) + (1 - v)(Q(X) + 2N(X))$ is the multiplicative identity of \mathcal{D}_2 . Then the generating idempotent for $\mathcal{E}_1 + (J(X))$ is $v(1 - N(X) + 2Q(X)) + (1 - v)(1 - Q(X) + 2N(X)) + J(X) - (v(1 - N(X) + 2Q(X)) + (1 - v)(1 - Q(X) + 2N(X)))J(X) = v(1 - N(X) + 2Q(X)) + (1 - v)(1 - Q(X) + 2N(X)) + J(X) + (J(X) - J(X)) = v(Q(X) + 2N(X)) + (1 - v)(N(X) + 2Q(X))$, which implies that $\mathcal{E}_1 + (J(X)) = \mathcal{D}_1$. Similarly, $\mathcal{E}_2 + (J(X)) = \mathcal{D}_2$.

- (vi) From Theorem 16, the generating idempotent for \mathcal{E}_1^\perp is $1 - (v(1 - N(X) + 2Q(X)) + (1 - v)(1 - Q(X) + 2N(X)))\mu_{-1} = v(N(X) + 2Q(X))\mu_{-1} + (1 - v)(Q(X) + 2N(X))\mu_{-1}$. Since $-1 \in N_p$ as $p \equiv -1 \pmod{8}$, it follows that $N(X)\mu_{-1} = Q(X)$ and $Q(X)\mu_{-1} = N(X)$. Therefore the generating idempotent for \mathcal{E}_1^\perp is $v(Q(X) + 2N(X)) + (1 - v)(N(X) + 2Q(X))$ implying that $\mathcal{E}_1^\perp = \mathcal{D}_1$. Similarly, $\mathcal{E}_2^\perp = \mathcal{D}_2$. From (v), we have $\mathcal{E}_i \subseteq \mathcal{D}_i$ implying that \mathcal{E}_i is self-orthogonal for $i = 1, 2$. \square

Case II: $p \equiv 1(\text{mod}8)$

Definition 6. Let $p - 1 = 8r$. If r is odd, define

$$\mathcal{D}_1 = (v(1 - N(X) + 2Q(X)) + (1 - v)(1 - Q(X) + 2N(X)))$$

$$\mathcal{D}_2 = (v(1 - Q(X) + 2N(X)) + (1 - v)(1 - N(X) + 2Q(X)))$$

and

$$\mathcal{E}_1 = (v(Q(X) + 2N(X)) + (1 - v)(N(X) + 2Q(X)))$$

$$\mathcal{E}_2 = (v(N(X) + 2Q(X)) + (1 - v)(Q(X) + 2N(X))).$$

If r is even, define

$$\mathcal{D}_1 = (v(1 + N(X)) + (1 - v)(1 + Q(X))),$$

$$\mathcal{D}_2 = (v(1 + Q(X)) + (1 - v)(1 + N(X))),$$

and

$$\mathcal{E}_1 = (v(-Q(X)) + (1 - v)(-N(X))),$$

$$\mathcal{E}_2 = (v(-N(X)) + (1 - v)(-Q(X))).$$

These eight cyclic codes of length p are called the quadratic residue codes over R at case II.

Similar to Theorem 17, we also have the following results. Here we omit the proof.

Theorem 18. Let $p \equiv 1(\text{mod}8)$. Then the quadratic residue codes defined above satisfy the following:

- (i) $\mathcal{D}_i \mu_a = \mathcal{D}_i$ and $\mathcal{E}_i \mu_a = \mathcal{E}_i$ for $i = 1, 2$ and $a \in \mathcal{Q}_p$; $\mathcal{D}_1 \mu_a = \mathcal{D}_2$ and $\mathcal{E}_1 \mu_a = \mathcal{E}_2$ for $a \in \mathcal{N}_p$.
- (ii) $\mathcal{D}_1 \cap \mathcal{D}_2 = (J(X))$ and $\mathcal{D}_1 + \mathcal{D}_2 = R_p$.
- (iii) $\mathcal{E}_1 \cap \mathcal{E}_2 = \{0\}$ and $\mathcal{E}_1 \cap \mathcal{E}_2 = (J(X))^\perp$.
- (iv) $|\mathcal{D}_1| = |\mathcal{D}_2| = 4^{p+1}$ and $|\mathcal{E}_1| = |\mathcal{E}_2| = 4^{p-1}$.
- (v) $\mathcal{D}_i = \mathcal{E}_i + (J(X))$ for $i = 1, 2$.
- (vi) $\mathcal{E}_1^\perp = \mathcal{D}_2$ and $\mathcal{E}_2^\perp = \mathcal{D}_1$.

Let \mathcal{D}_1 and \mathcal{D}_2 be the quadratic residue codes defined above. In the following, we discuss two extensions of \mathcal{D}_i denoted as $\widehat{\mathcal{D}}_i$ and $\widetilde{\mathcal{D}}_i$.

Definition 7. Let G_i be the generator matrix for the quadratic residue codes \mathcal{E}_i . Then we define $\widehat{\mathcal{D}}_i$ and $\widetilde{\mathcal{D}}_i$ with \widehat{G}_i and \widetilde{G}_i as their generator matrices as follows, respectively.

(i) If $p \equiv -1(\text{mod}8)$, then

$$\widehat{G}_i = \begin{bmatrix} 3 & 3 & \cdots & 3 \\ 0 & & & \\ \vdots & & G_i & \\ 0 & & & \end{bmatrix} \text{ and } \widetilde{G}_i = \begin{bmatrix} 1 & 3 & \cdots & 3 \\ 0 & & & \\ \vdots & & G_i & \\ 0 & & & \end{bmatrix}.$$

(ii) If $p \equiv 1(\text{mod}8)$, then

$$\widehat{G}_i = \begin{bmatrix} 3 & 1 & \cdots & 1 \\ 0 & & & \\ \vdots & & G_i & \\ 0 & & & \end{bmatrix} \text{ and } \widetilde{G}_i = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & & & \\ \vdots & & G_i & \\ 0 & & & \end{bmatrix}.$$

Theorem 19. Let \mathcal{D}_i be the quadratic residue codes of length p over R . The following hold

- (i) If $p \equiv -1(\text{mod}8)$, then $\widehat{\mathcal{D}}_i$ and $\widetilde{\mathcal{D}}_i$ are self-dual.
- (ii) If $p \equiv 1(\text{mod}8)$, then $\widehat{\mathcal{D}}_1^\perp = \widetilde{\mathcal{D}}_2$ and $\widehat{\mathcal{D}}_2^\perp = \widetilde{\mathcal{D}}_1$.

Proof. If $p \equiv -1(\text{mod}8)$, by the fact that the sum of the components of any codeword in \mathcal{E}_i is zero, we have $\widehat{\mathcal{D}}_i$ and $\widetilde{\mathcal{D}}_i$ are self-orthogonal. Furthermore, $|\mathcal{D}_i| = |\widehat{\mathcal{D}}_i| = |\widetilde{\mathcal{D}}_i| = 4^{p+1}$ implying $\widehat{\mathcal{D}}_i$ and $\widetilde{\mathcal{D}}_i$ are self-dual.

If $p \equiv 1(\text{mod}8)$, then $\mathcal{E}_1^\perp = \mathcal{D}_2$ and $\mathcal{E}_2^\perp = \mathcal{D}_1$. Hence the extended codewords arising from \mathcal{E}_i are orthogonal to all codewords in either $\widehat{\mathcal{D}}_j$ and $\widetilde{\mathcal{D}}_j$ where $j \neq i$. Since the product of the vectors $(3, 1, \dots, 1)$ and $(1, 1, \dots, 1)$ is $3 + p \equiv 0(\text{mod}4)$, we have $\widehat{\mathcal{D}}_j^\perp \subseteq \mathcal{D}_i$ where $j \neq i$. Furthermore, $|\mathcal{D}_i| = |\widehat{\mathcal{D}}_i| = |\widetilde{\mathcal{D}}_i| = 4^{p+1}$ implying $\widehat{\mathcal{D}}_j^\perp = \widetilde{\mathcal{D}}_i$ where $j \neq i$. \square

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